# Comment on "A generalized Helmholtz theorem for time-varying vector fields," by Artice M. Davis [Am. J. Phys. 74, 72–76 (2006)]

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In a recent paper Davis formulated the following generalization of the Helmholtz theorem for a time-varying vector field:[1]

$$\boldsymbol{F} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial \boldsymbol{A}}{\partial t}\right) + \nabla\times(\nabla\times\boldsymbol{A}), \tag{1}$$

where $\phi$ and $\boldsymbol{A}$ are the Lorenz gauge retarded potentials. The purposes of this comment are to point out that Davis's generalization is a version of the generalization of the Helmholtz theorem formulated some years ago by McQuistan[2] and Jefimenko[3] and more recently by the present author[4–6] and to show that Davis's expression for the field $\boldsymbol{F}$ is also valid for potentials in gauges other than the Lorenz gauge.

The generalized Helmholtz theorem states that a retarded vector field vanishing at infinity can be written as[4]

$$\boldsymbol{F} = -\nabla\int d^3x'\,\frac{[\nabla'\cdot\boldsymbol{F}]}{4\pi R} + \nabla\times\int d^3x'\,\frac{[\nabla'\times\boldsymbol{F}]}{4\pi R} + \frac{1}{c^2}\frac{\partial}{\partial t}\int d^3x'\,\frac{[\partial\boldsymbol{F}/\partial t]}{4\pi R}, \tag{2}$$

where the square brackets denote the retardation symbol, $R = |\mathbf{x}-\mathbf{x}'|$, and the integrals are over all space. If we define the potentials $\Phi$, $\mathbf{A}$, and $\mathbf{C}$ by

$$\Phi = \int d^3x'\,\frac{[\nabla'\cdot\boldsymbol{F}]}{4\pi R}, \tag{3a}$$

$$\mathbf{A} = \int d^3x'\,\frac{[\nabla'\times\boldsymbol{F}]}{4\pi R}, \tag{3b}$$

$$\mathbf{C} = \int d^3x'\,\frac{[\partial\boldsymbol{F}/\partial t]}{4\pi R}, \tag{3c}$$

then Eq. (2) can be written compactly as

$$\boldsymbol{F} = -\nabla\Phi + \nabla\times\mathbf{A} + \frac{1}{c^2}\frac{\partial\mathbf{C}}{\partial t}. \tag{4}$$

The potentials $\Phi$ and $\mathbf{A}$ in this formulation of the theorem are different from the potentials $\phi$ and $\boldsymbol{A}$ in Davis's formulation. It is not difficult to derive the relations $\partial\phi/\partial t = -c^2\Phi$, $\nabla\times\boldsymbol{A} = \mathbf{A}$, and $\partial\boldsymbol{A}/\partial t = \mathbf{C}$, which imply the formal equivalence between the two formulations.

The standard Helmholtz theorem is usually applied to solve the equations of electrostatics and magnetostatics. The generalization of this theorem[4] can be used to solve Maxwell's equations. The generalization proposed by Davis[1] can be used to elucidate the form of Maxwell's equations. The two versions of the generalized Helmholtz theorem are complementary.

In Ref. 1 the electric field $-\nabla\phi - \partial \boldsymbol{A}/\partial t$ and the magnetic field $\nabla \times \boldsymbol{A}$ are expressed in terms of the Lorenz gauge potentials, which were used to formulate Eq. (1) for the time-varying vector field $\boldsymbol{F}$. Equation (1) can also be formulated using potentials in other gauges. For example, it can be formulated for potentials in the velocity gauge[7] $\nabla\cdot\boldsymbol{A}+(1/v^2)\partial\phi/\partial t = 0$, a class of gauges containing the Coulomb gauge ($v = \infty$), the Lorentz gauge ($v = c$), and the Kirchhoff gauge[8] ($v = ic$). Jackson[7] recently derived the gauge function $\chi_v$ (Eq. (7.5) of Ref. 7), which transforms the Lorenz gauge potentials $\phi_L$ and $\boldsymbol{A}_L$ to the velocity gauge potentials $\phi_v$ and $\boldsymbol{A}_v$:

$$\phi_v = \phi_L - \frac{\partial \chi_v}{\partial t}, \tag{5a}$$

$$\boldsymbol{A}_v = \boldsymbol{A}_L + \nabla\chi_v. \tag{5b}$$

From Eqs. (5) we obtain

$$\nabla\phi_L + \frac{\partial \boldsymbol{A}_L}{\partial t} = \nabla\phi_v + \frac{\partial \boldsymbol{A}_v}{\partial t} \tag{6a}$$

$$\nabla\times\boldsymbol{A}_L = \nabla\times\boldsymbol{A}_v. \tag{6b}$$

Equations (6) imply that Eq. (1) is also valid for potentials in the velocity gauge, which means that it is valid for the Coulomb and Kirchhoff gauges also. The application of Eq. (1) to potentials in the velocity gauge requires the identification $\boldsymbol{F} = \mu_0\mathbf{J}$, where $\mathbf{J}$ is the current density and $\mu_0$ the permeability of free space.

Davis introduced causality in Eq. (1) when he chose retarded potentials. But we can equally choose acausal advanced potentials to obtain Eq. (1). Causality in Eq. (1) is not a necessary assumption, but it is required to identify $-\nabla\phi - \partial\boldsymbol{A}/\partial t$ and $\nabla\times\boldsymbol{A}$ with the retarded electric and magnetic fields. As pointed out by Rohrlich,[9] causality must be inserted by hand in classical field theories as a condition.

The reader might wonder why Eq. (1) can also be written in terms of the Coulomb gauge potentials when the instantaneous scalar potential $\phi_C$ in this gauge is clearly acausal. The explanation is that the Coulomb gauge vector potential $\mathbf{A}_C$ contains two parts, one of which is causal (retarded) and the other is acausal (instantaneous). Jackson[7] recently derived a novel expression for $\mathbf{A}_C$ (Eq. (3.10) in Ref. 7) which exhibits both parts. The fact that $\mathbf{A}_C$ carries a causality-violating instantaneous component has also been recently emphasized by Yang.[10] The effect of the acausal part of $\mathbf{A}_C$ vanishes identically when we take the curl and obtain $\nabla\times\boldsymbol{A}_C = \nabla\times\boldsymbol{A}_L$. A direct calculation gives[7] $-\partial\boldsymbol{A}_C/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t + \nabla\phi_C$.

The last (acausal) term cancels exactly the instantaneous electric field $-\nabla\phi_C$ generated by $\phi_C$ and we again obtain $-\nabla\phi_C - \partial\boldsymbol{A}_C/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t$. This expression has also been recently demonstrated in Ref. 11 using a different approach (see Eq. (29) in Ref. 11). In other words, the explicit presence of an acausal term in Eq. (1) when it is written in terms of the Coulomb gauge potentials is irrelevant because such a term is always canceled, which means that causality is never effectively lost.

Similar conclusions can be drawn when Eq. (1) is expressed in terms of the Kirchhoff gauge potentials $\phi_K$ and $\mathbf{A}_K$.[8] In this case the potential $\phi_K$ propagates with the imaginary speed $ic$ and generates the imaginary field $-\nabla\phi_K$. The Kirchhoff gauge vector potential $\mathbf{A}_K$ contains three parts: one is causal (retarded), one is imaginary, and the remaining one mixes imaginary and retarded contributions (see Eq. (42) in Ref. 8). The effect of the imaginary terms in the last two parts vanishes identically when we take the curl and obtain $\nabla\times\boldsymbol{A}_K = \nabla\times\boldsymbol{A}_L$. A direct calculation gives[8] $-\partial\boldsymbol{A}_K/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t + \nabla\phi_K$. The last term cancels exactly the imaginary field $-\nabla\phi_K$ and we again obtain $-\nabla\phi_K - \partial\boldsymbol{A}_K/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t$. The explicit presence of an imaginary term in Eq. (1) when it is written in terms of the Kirchhoff gauge potentials is irrelevant because such a term is always canceled, which means that causality is never effectively lost.

In the same sense that the Helmholtz theorem is considered as the mathematical foundation of electrostatics and magnetostatics, the generalized Helmholtz theorem can be considered as the mathematical foundation of electromagnetism. I advocate the use of both formulations of the generalized Helmholtz theorem[12] in courses of electromagnetism and invite instructors to decide which formulation they find more useful.

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1 A. M. Davis, "A generalized Helmholtz theorem for time-varying vector fields," Am. J. Phys. **74**, 72–76 (2006), Eq. (29).

2 R. B. McQuistan, *Scalar and Vector Fields: A Physical Interpretation* (Wiley, New York, 1965), Sec. 12-3, Eq. (12.37).

3 O. D. Jefimenko, *Electricity and Magnetism* (Electret Scientific, Star City, WV, 1989), 2nd ed., Sec. 2-14, Eq. (2-14.2).

[4] J. A. Heras, “Jefimenko's formulas with magnetic monopoles and the Lienard-Wiechert fields of a dual-charged particle,” Am. J. Phys. **62**, 525–531 (1994), Eq. (13).

[5] J. A. Heras, “Time-dependent generalizations of the Biot-Savart and Coulomb laws: A formal derivation,” Am. J. Phys. **63**, 928–932 (1995), Eq. (17).

[6] J. A. Heras, “Comment on ‘Causality, the Coulomb field, and Newton's law of gravitation’ by F. Rohrlich [Am. J. Phys. **70**, 411–414 (2002)],” Am. J. Phys. **71**, 729–730 (2003), Eqs. (5)–(7).

[7] J. D. Jackson, “From Lorenz to Coulomb and other explicit gauge transformations,” Am. J. Phys. **70**, 917–928 (2002).

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[9] F. Rohrlich, “Causality, the Coulomb field, and Newton's law of gravitation,” Am. J. Phys. **70**, 411–414 (2002).

[10] K.-H. Yang, “The physics of gauge transformations,” Am. J. Phys. **73**, 742–751 (2005).

[11] J. A. Heras, “Instantaneous fields in classical electrodynamics,” Europhys. Lett. **69**, 1–7 (2005).

[12] The author thanks V. Hnizdo for drawing his attention to the fact that both formulations of the generalized theorem are equivalent.